

The $J_{c,p,\lambda}^{m,\delta}$ integral operator

Reem A. Hamdan

Department of Mathematics , Palestine Polytechnic University , Hebron , Palestine

reemowaidat@yahoo.com

Amjad S. Barham

Dean of applied science college

Palestine Polytechnic University , Hebron , Palestine

amjad@ppu.edu

ABSTRACT

In this paper we introduce some new subclasses of strongly close-to-convex P-valent functions defined by a multiplier operator using the Komatu integral operator and study their inclusion relationships with the integral preserving properties.

1. INTRODUCTION

Let $A(p,n)$ be the class of functions $f(z)$ of the form

$$f(z) = z^p + \sum_{k=p+n}^{\infty} a_k z^k, \quad (p, n \in \mathbb{C}) \quad (1.1)$$

which are analytic and p- valent in the open unit disk $U = \{z : z \in \mathbb{C}, |z| < 1\}$

The generalized Komatu integral operator $K_{c,p}^{\delta} : A(p,n) \rightarrow A(p,n)$ is defined for $\delta > 0$ and $c > -p$ as

$$K_{c,p}^{\delta} f(z) = \frac{(c+p)^{\delta}}{\Gamma(\delta)} \int_0^z t^{c-1} (\log \frac{z}{t})^{\delta-1} f(t) dt \quad (1.2)$$

now, in terms of $K_{c,p}^{\lambda}$, we introduce the linear multiplier operator

$J_{c,p,\lambda}^{m,\delta} : A(p,n) \rightarrow A(p,n)$ as follows

$$J_{c,p,\lambda}^{0,0} f(z) = f(z) \quad (1.3)$$

$$\begin{aligned} J_{c,p,\lambda}^{1,\delta} f(z) &= (1-\lambda) K_{c,p}^{\delta} f(z) + \frac{\lambda z}{p} (K_{c,p}^{\delta} f(z))' = J_{c,p,\lambda}^{\delta} f(z) \\ &\vdots \\ J_{c,p,\lambda}^{m,\delta} f(z) &= J_{c,p,\lambda}^{\delta} (J_{c,p,\lambda}^{m-1,\delta} f(z)) \end{aligned}$$

for $\delta > 0, c > -p, \lambda \geq 0$, and $m \in \mathbb{C}$.

If $f \in A(p,n)$ is given by (1.1), then

$$J_{c,p,\lambda}^{m,\delta} f(z) = z^p + \sum_{k=p+n}^{\infty} B_{k,m}(c, p, \lambda, \delta) a_k z^k \quad (1.4)$$

Where

$$B_{k,m}(c,p,\lambda,\delta) = \left[\left(\frac{c+p}{c+k} \right)^\delta \left(1 + \frac{\lambda}{p} (k-p) \right) \right]^m \quad (1.5)$$

for $\delta > 0, c > -p, \lambda \geq 0$, and $m \in \mathbb{C}$.

If $f(z)$ and $g(z)$ are analytic in U , we say that $f(z)$ is subordinate to $g(z)$, written $f \prec g$ or $f(z) \prec g(z)$, if there exists a Schwarz function $w(z)$ in U such that $f(z) = g(w(z))$.

Let $S_{c,p,\lambda}^{m,\delta}(\eta, A, B)$ be the class of functions $f \in A(p, n)$ satisfying the condition

$$\frac{1}{p-\eta} \left(\frac{z(J_{c,p,\lambda}^{m,\delta} f'(z))'}{J_{c,p,\lambda}^{m,\delta} f(z)} - \eta \right) \prec \frac{1+Az}{1+Bz} \quad (1.6)$$

2. Main Results

Lemma 1. Let $h(z)$ be convex univalent in U with $h(0)=1$ and

$\operatorname{Re}\{vh(z)+\mu\} > 0$ ($v, \mu \in \mathbb{C}$). If $p(z)$ is analytic in U with $p(0)=1$, then

$$p(z) + \frac{zp'(z)}{vp(z)+\mu} \prec h(z), \quad (z \in U), \text{ implies } p(z) \prec h(z), \quad (z \in U)$$

Lemma 2. Let $h(z)$ be convex univalent in U and $w(z)$ be analytic in U with

$\operatorname{Re}w(z) \geq 0$. If $p(z)$ is analytic in U with $p(0)=h(0)$, then $p(z)+w(z)zp'(z) \prec h(z)$, $(z \in U)$ implies $p(z) \prec h(z)$, $(z \in U)$.

Lemma 3. Let $p(z)$ be analytic in U with $p(0)=1$ and $p(z) \neq 0$ in U . If there exist two points $z_1, z_2 \in U$ such that

$$-\frac{\pi}{2} \alpha_1 = \arg p(z_1) < \arg p(z) < \arg p(z_2) = \frac{\pi}{2} \alpha_2 \quad (2.1)$$

for some α_1, α_2 ($\alpha_1, \alpha_2 > 0$) and for all ($|z| < |z_1| = |z_2|$), then we have

$$\frac{z_1 p'(z_1)}{p(z_1)} = -i \frac{(\alpha_1 + \alpha_2)}{2} m \quad \text{and} \quad \frac{z_2 p'(z_2)}{p(z_2)} = i \frac{(\alpha_1 + \alpha_2)}{2} m \quad (2.2)$$

where

$$m \geq \frac{1 - |\mathcal{C}|^*}{1 + |\mathcal{C}|^*} \quad \text{and} \quad \mathcal{C}^* = i \tan \frac{\pi}{4} \left(\frac{\alpha_2 - \alpha_1}{\alpha_2 + \alpha_1} \right) \quad (2.3)$$

Proposition 1. Let $\delta > 0, c > -p, \lambda \geq 0$, $m \in \mathbb{C}$ and $h(z)$ be convex univalent in U with $h(0)=1$ and $\operatorname{Re}h(z) > 0$. If a function $f(z) \in A(p, n)$ satisfies the condition

$$\frac{1}{p-\eta} \left(\frac{z \left(J_{c,p,\lambda}^{m,\delta-\frac{1}{m}} f'(z) \right)'}{J_{c,p,\lambda}^{m,\delta-\frac{1}{m}} f(z)} - \eta \right) \prec h(z), \quad (0 \leq \eta < 1; z \in U), \text{ then}$$

$$\frac{1}{p-\eta} \left(\frac{z \left(J_{c,p,\lambda}^{m,\delta} f(z) \right)'}{J_{c,p,\lambda}^{m,\delta} f(z)} - \eta \right) \prec h(z), \quad (0 \leq \eta < 1; z \in U).$$

Proof:

Let

$$d(z) = \frac{1}{p-\eta} \left(\frac{z \left(J_{c,p,\lambda}^{m,\delta} f(z) \right)'}{J_{c,p,\lambda}^{m,\delta} f(z)} - \eta \right) \quad (2.4)$$

where $d(z)$ is analytic function in U

$$\begin{aligned} d(z) &= \frac{1}{p-\eta} \left(\frac{(c+p) J_{c,p,\lambda}^{m,\delta-\frac{1}{m}} f(z) - c J_{c,p,\lambda}^{m,\delta} f(z)}{J_{c,p,\lambda}^{m,\delta} f(z)} - \eta \right) \\ &= \frac{1}{p-\eta} \left(\frac{(c+p) J_{c,p,\lambda}^{m,\delta-\frac{1}{m}} f(z)}{J_{c,p,\lambda}^{m,\delta} f(z)} - c - \eta \right) \\ &= \frac{1}{p-\eta} \left(\frac{c z^p + p z^p + \sum_{k=p+n}^{\infty} a_k z^k \frac{(c+p)^{\delta}}{(c+k)^{\delta m-1}} (1+(k-p)\frac{\lambda}{p})^m}{z^p + \sum_{k=p+n}^{\infty} \left[\left(\frac{c+p}{c+k} \right)^{\delta} (1+\frac{\lambda}{p}(k-p)) \right] a_k z^k} - c - \eta \right) \end{aligned}$$

then $d(0)=1$

From (2.4)

$$(p-\eta)d(z)+c+\eta = \frac{(c+p) J_{c,p,\lambda}^{m,\delta-\frac{1}{m}} f(z)}{J_{c,p,\lambda}^{m,\delta} f(z)} \quad (2.5)$$

Take the logarithm to both sides

$$\ln((p-\eta)d(z)+c+\eta) = \ln(c+p) J_{c,p,\lambda}^{m,\delta-\frac{1}{m}} f(z) - \ln J_{c,p,\lambda}^{m,\delta} f(z)$$

Differentiating both sides with respect to z and multiply both sides by z

$$\frac{z(p-\eta)d'(z)}{(p-\eta)d(z)+c+\eta} = \frac{z(J_{c,p,\lambda}^{m,\delta-\frac{1}{m}} f(z))'}{J_{c,p,\lambda}^{m,\delta-\frac{1}{m}} f(z)} - \frac{z(J_{c,p,\lambda}^{m,\delta} f(z))'}{J_{c,p,\lambda}^{m,\delta} f(z)} \quad (2.6)$$

Dividing both sides by $p-\eta$

$$\frac{zd'(z)}{(p-\eta)d(z)+c+\eta} + d(z) = \frac{1}{p-\eta} \left[\frac{z(J_{c,p,\lambda}^{m,\delta-\frac{1}{m}} f(z))'}{J_{c,p,\lambda}^{m,\delta-\frac{1}{m}} f(z)} - \eta \right] \quad (2.7)$$

By using lemma (1) , it follows that $d(z) \prec h(z), (z \in U)$, then

$$\frac{1}{p-\eta} \left(\frac{z \left(J_{c,p,\lambda}^{m,\delta} f(z) \right)'}{J_{c,p,\lambda}^{m,\delta} f(z)} - \eta \right) \prec h(z)$$

Proposition 2. Let $h(z)$ be a convex univalent in U with $h(0)=1$ and $\operatorname{Re}(h(z))>0$.

If a function $f(z) \in A(p,n)$ satisfies the condition

$$\frac{1}{p-\eta} \left(\frac{z \left(J_{c,p,\lambda}^{m,\delta} f(z) \right)'}{J_{c,p,\lambda}^{m,\delta} f(z)} - \eta \right) \prec h(z), \quad (0 \leq \eta < 1; z \in U), \text{ then}$$

$$\frac{1}{p-\eta} \left(\frac{z \left(J_{c,p,\lambda}^{m,\delta} L_f(z) \right)'}{J_{c,p,\lambda}^{m,\delta} L_f(z)} - \eta \right) \prec h(z), \quad (0 \leq \eta < 1; z \in U),$$

where $L_\theta(f)$ is the integral operator defined by

$$L_\theta f = L_f(z) = \frac{(\theta+1)}{z} \int_0^z t^{\theta-1} f(t) dt, \quad (\theta \geq 0) \quad (2.8)$$

Proof:

$$f(z) = z^p + \sum_{k=p+n}^{\infty} a_k z^k \quad (p, n \in \mathbb{N})$$

Then

$$\begin{aligned} L_\theta f(z) &= \frac{(\theta+1)}{z} \left[\int_0^z t^{\theta-1} t^p dt + \int_0^z t^{\theta-1} \sum_{k=p+n}^{\infty} a_k t^k dt \right] \\ L_\theta f(z) &= \left(\frac{\theta+1}{\theta+p} \right) z^p + \sum_{k=p+n}^{\infty} a_k z^k \left(\frac{\theta+1}{\theta+k} \right) \end{aligned} \quad (2.9)$$

Now

$$\begin{aligned} k_{c,p}^\delta L_\theta f(z) &= \frac{(c+p)}{\Gamma(\delta) z} \int_0^z t^{c-1} \left(\log \frac{z}{t} \right)^{\delta-1} L_f(t) dt \\ &= \frac{(c+p)}{\Gamma(\delta) z} \int_0^z t^{c-1} \left(\log \frac{z}{t} \right)^{\delta-1} \left(\frac{\theta+1}{\theta+p} t^p dt + \sum_{k=p+n}^{\infty} a_k \left(\frac{\theta+1}{\theta+k} \right) \int_0^z t^{c-1} \left(\log \frac{z}{t} \right)^{\delta-1} t^k dt \right) \end{aligned}$$

then

$$k_{c,p}^\delta L_f(z) = \frac{\theta+1}{\theta+p} z^p + \sum_{k=p+n}^{\infty} \left(\frac{\theta+1}{\theta+k} \right) \left(\frac{c+p}{c+k} \right)^\delta a_k z^k \quad (2.10)$$

and

$$(k_{c,p}^\delta L_f(z))' = p \frac{\theta+1}{\theta+p} z^{p-1} + \sum_{k=p+n}^{\infty} \left(\frac{\theta+1}{\theta+k} \right) \left(\frac{c+p}{c+k} \right)^\delta a_k k z^{k-1}$$

then

$$\begin{aligned} J_{c,p,\lambda}^{1,\delta} (L_\theta f(z)) &= (1-\lambda) k_{c,p}^\delta (L_\theta f(z)) + \frac{\lambda z}{p} (k_{c,p}^\delta L_\theta f(z))' \\ J_{c,p,\lambda}^{1,\delta} (L_f(z)) &= \frac{\theta+1}{\theta+p} z^p + \sum_{k=p+n}^{\infty} a_k z^k \left(\frac{\theta+1}{\theta+k} \right) \left(\frac{c+p}{c+k} \right)^\delta [1 + \frac{\lambda}{p} (k-p)] \end{aligned}$$

By induction

$$J_{c,p,\lambda}^{m,\delta} (L_f(z)) = \frac{\theta+1}{\theta+p} z^p + \sum_{k=p+n}^{\infty} \left(\frac{\theta+1}{\theta+k} \right) \left(\frac{c+p}{c+k} \right)^\delta [1 + \frac{\lambda}{p} (k-p)]^m a_k z^k \quad (2.11)$$

and

$$z (J_{c,p,\lambda}^{m,\delta} L_f(z))' = (\theta+1) J_{c,p,\lambda}^{m,\delta} f(z) - \theta J_{c,p,\lambda}^{m,\delta} L_f(z)$$

Let

$$d(z) = \frac{1}{p-\eta} \left(\frac{z(J_{c,p,\lambda}^{m,\delta} L_\theta f(z))'}{J_{c,p,\lambda}^{m,\delta} L_\theta f(z)} - \eta \right), \quad (z \in U) \quad (2.12)$$

where $d(z)$ is analytic function in U , with $d(0)=1$

Now

$$(p-\eta)d(z)+\eta=\frac{z(J_{c,p,\lambda}^{m,\delta} L_\theta f(z))'}{J_{c,p,\lambda}^{m,\delta} L_\theta f(z)},$$

$$(p-\eta)d(z)+\eta+\theta=(\theta+1)\frac{J_{c,p,\lambda}^{m,\delta} f(z)}{J_{c,p,\lambda}^{m,\delta} L f(z)}$$

Differentiating logarithmically with respect to z and multiply by z

$$\frac{z(p-\eta)d'(z)}{\theta+\eta+(p-\eta)d(z)}+[(p-\eta)d+\eta]=\frac{z(J_{c,p,\lambda}^{m,\delta} f(z))'}{J_{c,p,\lambda}^{m,\delta} f(z)}$$

Dividing both sides by $(p-\eta)$ and add and subtract $\frac{\eta}{p-\eta}$

$$\frac{zd'(z)}{\theta+\eta+(p-\eta)d(z)}+d(z)=\frac{1}{p-\eta}\left[\frac{z(J_{c,p,\lambda}^{m,\delta} f(z))'}{J_{c,p,\lambda}^{m,\delta} f(z)}-\eta\right], \quad z \in U$$

Therefore by lemma (1), we obtain

$$d(z)+\frac{zd'(z)}{c+\eta+(p-\eta)d(z)}\prec h(z)$$

Then

$$d(z)=\frac{1}{p-\eta}\left[\frac{z(J_{c,p,\lambda}^{m,\delta} (L_\theta f(z)))'}{J_{c,p,\lambda}^{m,\delta} (L_\theta f(z))}-\eta\right]\prec h(z)$$

Corollary 1. If $f(z) \in S_{c,p,\lambda}^{m,\delta}(\eta, A, B)$, then $L_\theta(f)$, where $L_\theta(f)$ is the integral operator defined by (2.8).

Proof:

In proposition (2), take $h(z)=\frac{1+Az}{1+Bz}$, then

$$\frac{1}{p-\eta}\left[\frac{z(J_{c,p,\lambda}^{m,\delta} f(z))'}{J_{c,p,\lambda}^{m,\delta} f(z)}-\eta\right]\prec\frac{1+Az}{1+Bz}$$

$$\frac{1}{p-\eta}\left[\frac{z(J_{c,p,\lambda}^{m,\delta} (L_\theta f(z)))'}{J_{c,p,\lambda}^{m,\delta} (L_\theta f(z))}-\eta\right]\prec\frac{1+Az}{1+Bz}$$

Proposition 3. let $h(z)$ be convex univalent in U with $h(0)=1$ and $\operatorname{Re}(h(z))>0$.

If a function $f(z) \in A(p,n)$ satisfies the condition

$$\frac{1}{p-\eta} \left(\frac{z \left(J_{c,p,\lambda}^{m,\delta-\frac{1}{m}} L_f(z) \right)' - \eta}{J_{c,p,\lambda}^{m,\delta-\frac{1}{m}} L_f(z)} \right) \prec h(z), \quad (0 \leq \eta < 1; z \in U)$$

Then

$$\frac{1}{p-\eta} \left(\frac{z \left(J_{c,p,\lambda}^{m,\delta} L_f(z) \right)' - \eta}{J_{c,p,\lambda}^{m,\delta} L_f(z)} \right) \prec h(z), \quad (0 \leq \eta < 1; z \in U)$$

Where $L_\theta(f)$ is the integral operator defined by (2.8).

Proof:

$$z (J_{c,p,\lambda}^{m,\delta} L_f(z))' = (c+p) J_{c,p,\lambda}^{m,\delta-\frac{1}{m}} L_f(z) - c J_{c,p,\lambda}^{m,\delta} L_f(z)$$

Let

$$d(z) = \frac{1}{p-\eta} \left(\frac{z (J_{c,p,\lambda}^{m,\delta} L_f(z))' - \eta}{J_{c,p,\lambda}^{m,\delta} L_f(z)} \right), \quad (z \in U) \quad (2.14)$$

Where $d(z)$ is analytic function in U , with $d(0)=1$

Now

$$(p-\eta)d(z) + \eta = \frac{z (J_{c,p,\lambda}^{m,\delta} L_f(z))'}{J_{c,p,\lambda}^{m,\delta} L_f(z)}$$

Differentiating logarithmically with respect to z and multiply by z

$$\frac{z(p-\eta)d'(z)}{c+\eta+(p-\eta)d(z)} = \frac{z (J_{c,p,\lambda}^{m,\delta-\frac{1}{m}} (L_f(z)))'}{J_{c,p,\lambda}^{m,\delta-\frac{1}{m}} (L_f(z))} - \frac{z (J_{c,p,\lambda}^{m,\delta} (L_f(z)))'}{J_{c,p,\lambda}^{m,\delta} (L_f(z))}$$

Dividing both sides by $(p-\eta)$ and add and subtract $\frac{\eta}{p-\eta}$

$$\frac{zd'(z)}{c+\eta+(p-\eta)d(z)} + d(z) = \frac{1}{p-\eta} \left[\frac{z (J_{c,p,\lambda}^{m,\delta-\frac{1}{m}} (L_f(z)))'}{J_{c,p,\lambda}^{m,\delta-\frac{1}{m}} (L_f(z))} - \eta \right]$$

Therefore by lemma (1), we obtain

$$d(z) + \frac{zd'(z)}{c+\eta+(p-\eta)d(z)} \prec h(z)$$

Then

$$d(z) = \frac{1}{p-\eta} \left[\frac{z (J_{c,p,\lambda}^{m,\delta} (L_f(z)))'}{J_{c,p,\lambda}^{m,\delta} (L_f(z))} - \eta \right] \prec h(z)$$

Theorem 1. Let $f(z) \in A(p,n)$ and $0 < \delta_1, \delta_2 \leq 1, 0 \leq \gamma < 1$, if

$$-\frac{\pi}{2} \delta_1 < \arg \left(\frac{z (J_{c,p,\lambda}^{m,\delta-\frac{1}{m}} f(z))'}{J_{c,p,\lambda}^{m,\delta-\frac{1}{m}} g(z)} - \gamma \right) < \frac{\pi}{2} \delta_2$$

for some $g(z) \in S_{c,p,\lambda}^{m,\delta}(\eta, A, B)$, then

$$-\frac{\pi}{2} \alpha_1 < \arg \left(\frac{z (J_{c,p,\lambda}^{m,\delta} f(z))'}{J_{c,p,\lambda}^{m,\delta} g(z)} - \gamma \right) < \frac{\pi}{2} \alpha_2$$

, where α_1, α_2 ($0 < \alpha_1, \alpha_2 \leq 1$), are the solutions of the equations :

$$\delta_1 = \begin{cases} \alpha_1 + \frac{2}{\pi} \tan^{-1} \left(\frac{(\alpha_1 + \alpha_2)(1 - |C^*|) \cos \frac{\pi}{2} t_1}{2 \left(\frac{(p-\eta)(1+A)}{1+B} + \eta + c \right) (1 + |C^*|) + (\alpha_1 + \alpha_2)(1 - |C^*|) \sin \frac{\pi}{2} t_1} \right) & \text{for } B \neq -1 \\ \alpha_1 & \text{for } B = -1 \end{cases} \quad (2.15)$$

and

$$\delta_2 = \begin{cases} \alpha_2 + \frac{2}{\pi} \tan^{-1} \left(\frac{(\alpha_1 + \alpha_2)(1 - |C^*|) \cos \frac{\pi}{2} t_1}{2 \left(\frac{(p-\eta)(1+A)}{1+B} + \eta + c \right) (1 + |C^*|) + (\alpha_1 + \alpha_2)(1 - |C^*|) \sin \frac{\pi}{2} t_1} \right) & \text{for } B \neq -1 \\ \alpha_2 & \text{for } B = -1 \end{cases} \quad (2.16)$$

where C^* is given by (2.3) and

$$t_1 = \frac{2}{\pi} \sin^{-1} \left(\frac{(p-\eta)(1-B)}{(p-\eta)(1-AB) + (\eta+c)(1-B^2)} \right) \quad (2.17)$$

Proof

Let

$$d(z) = \frac{1}{p-\gamma} \left(\frac{z(J_{c,p,\lambda}^{m,\delta} f(z))'}{J_{c,p,\lambda}^{m,\delta} g(z)} - \gamma \right), \quad (z \in U)$$

Now

$$z(J_{c,p,\lambda}^{m,\delta} f(z))' = (c+p) J_{c,p,\lambda}^{m,\delta - \frac{1}{m}} f(z) - c J_{c,p,\lambda}^{m,\delta} f(z)$$

and

$$((p-\gamma)d(z) + \gamma) J_{c,p,\lambda}^{m,\delta} g(z) = z(J_{c,p,\lambda}^{m,\delta} f(z))' = (c+p) J_{c,p,\lambda}^{m,\delta - \frac{1}{m}} f(z) - c J_{c,p,\lambda}^{m,\delta} f(z)$$

Differentiating both sides with respect to z and multiply by z

$$\begin{aligned} & z(p-\gamma)d'(z) J_{c,p,\lambda}^{m,\delta} g(z) + z[(p-\gamma)d(z) + \gamma](J_{c,p,\lambda}^{m,\delta} g(z))' \\ &= z(c+p)(J_{c,p,\lambda}^{m,\delta - \frac{1}{m}} f(z))' - zc(J_{c,p,\lambda}^{m,\delta} f(z))' \end{aligned} \quad (2.18)$$

Let

$$q(z) = \frac{1}{p-\eta} \left(\frac{z(J_{c,p,\lambda}^{m,\delta} g(z))'}{J_{c,p,\lambda}^{m,\delta} g(z)} - \eta \right), \quad (z \in U)$$

Then

$$\begin{aligned} (p-\eta)q + \eta &= \frac{z(J_{c,p,\lambda}^{m,\delta} g(z))'}{J_{c,p,\lambda}^{m,\delta} g(z)} = \frac{(c+p) J_{c,p,\lambda}^{m,\delta - \frac{1}{m}} g(z)}{J_{c,p,\lambda}^{m,\delta} g(z)} - c \\ (p-\eta)q + \eta + c &= \frac{(c+p) J_{c,p,\lambda}^{m,\delta - \frac{1}{m}} g(z)}{J_{c,p,\lambda}^{m,\delta} g(z)} \end{aligned} \quad (2.19)$$

From (2.18) and (2.19)

$$\begin{aligned} & z(p-\gamma)d'(z) J_{c,p,\lambda}^{m,\delta} g(z) + z[(p-\gamma)d(z) + \gamma](J_{c,p,\lambda}^{m,\delta} g(z))' \\ &= z(c+p)(J_{c,p,\lambda}^{m,\delta - \frac{1}{m}} f(z))' - zc(J_{c,p,\lambda}^{m,\delta} f(z))' \end{aligned}$$

Dividing both sides by $(p-\gamma) J_{c,p,\lambda}^{m,\delta} g(z)$

$$zd'(z) + \frac{[(p-\gamma)d+\gamma]}{(p-\gamma)}((p-\eta)q+\eta) = \frac{z(c+p)}{(p-\gamma)} \frac{(J_{c,p,\lambda}^{m,\delta-\frac{1}{m}}f(z))'}{J_{c,p,\lambda}^{m,\delta}g(z)} - \frac{[(p-\gamma)d(z)+\gamma]c}{(p-\gamma)}$$

$$zd'(z) + \frac{[(p-\gamma)d+\gamma]}{(p-\gamma)}((p-\eta)q+\eta+c) = \frac{z(c+p)}{(p-\gamma)} \frac{(J_{c,p,\lambda}^{m,\delta-\frac{1}{m}}f(z))'}{J_{c,p,\lambda}^{m,\delta}g(z)}$$

Dividing both sides by $[(p-\eta)q+\eta+c]$

$$\frac{zd'(z)}{(p-\eta)q+\eta+c} + d(z) = \frac{1}{(p-\gamma)} \left[\frac{z(J_{c,p,\lambda}^{m,\delta-\frac{1}{m}}f(z))'}{J_{c,p,\lambda}^{m,\delta-\frac{1}{m}}g(z)} - \gamma \right]$$

While , by using the result of Silverman and Silvia [11] , we have

$$\left| q(z) - \frac{1-AB}{1-B^2} \right| < \frac{(A-B)}{1-B^2}, (z \in U; B \neq -1) \quad (2.20)$$

and

$$\operatorname{Re}\{q(z)\} > \frac{1-A}{2}, \quad (z \in U; B = -1) \quad (2.21)$$

Then from (2.20) and (2.21), we obtain

$$(p-\eta)q+\eta+c = \rho e^{\frac{i\pi\phi}{2}}$$

Where

$$\begin{cases} \frac{(p-\eta)(1-A)}{(1-B)} + \eta + c < \rho < \frac{(p-\eta)(1+A)}{(1+B)} + \eta + c \\ -t_1 < \phi < t_1 \end{cases} \quad \text{for } B \neq -1$$

Where t_1 is given by (2.17), and

$$\begin{cases} \frac{(p-\eta)(1-A)}{2} + \eta + c < \rho < \infty \\ -1 < \phi < 1 \end{cases} \quad \text{for } B = -1$$

Here we note that $d(z)$ is analytic in U with $d(0)=1$ and $\operatorname{Re}(d(z))>0$ in U , by applying the assumption and lemma (2) with $w(z) = \frac{1}{(p-\eta)q+\eta+c}$.

Here $d(z) \neq 0$ in U . If there exist two points $z_1, z_2 \in U$ such that the condition (2.1) is satisfied , then(by lemma 3) , we obtain (2.2) under the restriction (2.3). At first , for the case $B \neq -1$, we obtain:

$$\arg(d(z_1) + \frac{z_1 d'(z_1)}{(p-\eta)q(z_1)+\eta+c}) \leq -\frac{\pi}{2} \alpha_1 - \tan^{-1} \left\{ \frac{(\alpha_1 + \alpha_2)(1-|c^*|)\cos\frac{\pi}{2}t_1}{2\left(\frac{(p-\eta)(1+A)}{(1+B)} + \eta + c\right)(1+|c^*|) + (\alpha_1 + \alpha_2)(1-|c^*|)\sin\frac{\pi}{2}t_1} \right\}$$

$$= -\frac{\pi}{2} \delta_1$$

and

$$\arg(d(z_2) + \frac{z_2 d'(z_2)}{(p-\eta)q(z_2) + \eta + c}) \geq \frac{\pi}{2} \alpha_2 + \tan^{-1} \left\{ \frac{(\alpha_1 + \alpha_2)(1 - |c|) \cos \frac{\pi}{2} t_1}{2 \left(\frac{(p-\eta)(1+A)}{(1+B)} + \eta + c \right) (1 + |c|) + (\alpha_1 + \alpha_2)(1 - |c|) \sin \frac{\pi}{2} t_1} \right\} = \frac{\pi}{2} \delta_2$$

Where we have used inequality (2.3), and δ_1, δ_2, t_1 are given by (2.15), (2.16) and (2.17), respectively.

Similarly, for the case $B = -1$, we obtain

$$\arg(d(z_1) + \frac{z_1 d'(z_1)}{(p-\eta)q(z_1) + \eta + c}) \leq -\frac{\pi}{2} \alpha_1$$

and

$$\arg(d(z_2) + \frac{z_2 d'(z_2)}{(p-\eta)q(z_2) + \eta + c}) \geq \frac{\pi}{2} \alpha_2$$

These are contradiction to the assumption of theorem (1), this completes the proof of theorem (1).

Corollary2. Let $f(z) \in A(p, n)$, if

$$\left| \arg \left(\frac{z(I_p^{\delta-1}f(z))'}{I_p^{\delta-1}g(z)} - \gamma \right) \right| < \frac{\pi}{2} \delta$$

Then

$$\left| \arg \left(\frac{z(I_p^{\delta}f(z))'}{I_p^{\delta}g(z)} - \gamma \right) \right| < \frac{\pi}{2} \alpha$$

where

$$I_p^{\delta-1}f(z) = J_{1,p,\theta}^{1,\delta-1}f(z) = z^p + \sum_{k=p+n}^{\infty} a_k z^k \left(\frac{1+p}{1+k} \right)^{\delta-1} \quad (2.22)$$

Proof:

In theorem (1), if we take $m = 1, c = 1, \lambda = 0, \delta_1 = \delta_2 = \delta$ and $\alpha_1 = \alpha_2 = \alpha$

$$I_p^{\delta-1}f(z) = J_{1,p,\theta}^{1,\delta-1}f(z) = z^p + \sum_{k=p+n}^{\infty} a_k z^k \left(\frac{1+p}{1+k} \right)^{\delta-1}$$

then

$$\left| \arg \left(\frac{z(I_p^{\delta-1}f(z))'}{I_p^{\delta-1}g(z)} - \gamma \right) \right| < \frac{\pi}{2} \delta$$

and

$$\left| \arg \left(\frac{z(I_p^{\delta}f(z))'}{I_p^{\delta}g(z)} - \gamma \right) \right| < \frac{\pi}{2} \alpha$$

Theorem 2. Let $f(z) \in A(p, n)$ and $0 < \delta_1, \delta_2 \leq 1, 0 \leq \gamma < 1$, if

$$-\frac{\pi}{2} \delta_1 < \arg\left(\frac{z(J_{c,p,\lambda}^{m,\delta} f(z))'}{J_{c,p,\lambda}^{m,\delta} g(z)} - \gamma\right) < \frac{\pi}{2} \delta_2$$

for some $g(z) \in S_{c,p,\lambda}^{m,\delta}(\eta, A, B)$, then

$$-\frac{\pi}{2} \alpha_1 < \arg\left(\frac{z(J_{c,p,\lambda}^{m,\delta} L_\theta f(z))'}{J_{c,p,\lambda}^{m,\delta} L_\theta g(z)} - \gamma\right) < \frac{\pi}{2} \alpha_2,$$

where $L_\theta(f)$ is defined by (2.8), α_1, α_2 ($0 < \alpha_1, \alpha_2 \leq 1$) is the solutions of the equations

$$\delta_1 = \begin{cases} \alpha_1 + \frac{2}{\pi} \tan^{-1}\left(\frac{(\alpha_1 + \alpha_2)(1 - |C^*|)\cos\frac{\pi}{2}t_2}{2\left(\frac{(p-\eta)(1+A)}{1+B} + \eta + \theta\right)(1 + |C^*|) + (\alpha_1 + \alpha_2)(1 - |C^*|)\sin\frac{\pi}{2}t_2}\right) & \text{for } B \neq -1 \\ \alpha_1 & \text{for } B = -1 \end{cases} \quad (2.23)$$

and

$$\delta_2 = \begin{cases} \alpha_2 + \frac{2}{\pi} \tan^{-1}\left(\frac{(\alpha_1 + \alpha_2)(1 - |C^*|)\cos\frac{\pi}{2}t_2}{2\left(\frac{(p-\eta)(1+A)}{1+B} + \eta + \theta\right)(1 + |C^*|) + (\alpha_1 + \alpha_2)(1 - |C^*|)\sin\frac{\pi}{2}t_2}\right) & \text{for } B \neq -1 \\ \alpha_2 & \text{for } B = -1 \end{cases} \quad (2.24)$$

where C^* is given by (2.3) and

$$t_2 = \frac{2}{\pi} \sin^{-1}\left(\frac{(p-\eta)(A-B)}{(p-\eta)(1-AB) + (\eta+\theta)(1-B^2)}\right) \quad (2.25)$$

Proof:

Let

$$d(z) = \frac{1}{p-\gamma} \left(\frac{z(J_{c,p,\lambda}^{m,\delta} L_\theta f(z))'}{J_{c,p,\lambda}^{m,\delta} L_\theta g(z)} - \gamma \right),$$

since $g(z) \in S_{c,p,\lambda}^{m,\delta}(\eta, A, B)$, then by corollary (1), $L_\theta(g) \in S_{c,p,\lambda}^{m,\delta}(\eta, A, B)$, then

$$[(p-\gamma)d(z) + \gamma](J_{c,p,\lambda}^{m,\delta} L_\theta g(z)) = z(J_{c,p,\lambda}^{m,\delta} L_\theta f(z))'$$

From proposition (2)

$$z(J_{c,p,\lambda}^{m,\delta} L_\theta f(z))' = (\theta+1)J_{c,p,\lambda}^{m,\delta} f(z) - \theta J_{c,p,\lambda}^{m,\delta} L_\theta f(z)$$

$$((p-\gamma)d + \gamma)(J_{c,p,\lambda}^{m,\delta} L_\theta g(z)) = (\theta+1)J_{c,p,\lambda}^{m,\delta} f(z) - \theta J_{c,p,\lambda}^{m,\delta} L_\theta f(z)$$

Differentiating both sides with respect to z

$$((p-\gamma)d + \gamma)(J_{c,p,\lambda}^{m,\delta} L_\theta g(z))' + (p-\gamma)d'(z)J_{c,p,\lambda}^{m,\delta} L_\theta g(z) = (\theta+1)(J_{c,p,\lambda}^{m,\delta} f(z))' - \theta(J_{c,p,\lambda}^{m,\delta} L_\theta f(z))'$$

Dividing both sides by $(J_{c,p,\lambda}^{m,\delta} L_\theta g(z))$ and multiply both sides by z , then

$$z((p-\gamma)d+\gamma)\frac{(J_{c,p,\lambda}^{m,\delta}L_\theta g(z))'}{J_{c,p,\lambda}^{m,\delta}L_\theta g(z)}+(p-\gamma)zd'(z)=z(\theta+1)\frac{(J_{c,p,\lambda}^{m,\delta}f(z))'}{J_{c,p,\lambda}^{m,\delta}L_\theta g(z)}-z\theta\frac{(J_{c,p,\lambda}^{m,\delta}L_\theta f(z))'}{J_{c,p,\lambda}^{m,\delta}L_\theta g(z)}$$

Let

$$q(z)=\frac{1}{p-\eta}\left(\frac{z(J_{c,p,\lambda}^{m,\delta}L_\theta g(z))'}{J_{c,p,\lambda}^{m,\delta}L_\theta g(z)}-\eta\right)$$

Then

$$((p-\gamma)d+\gamma)(q(p-\eta)+\eta+\theta)+(p-\gamma)zd'(z)=z(\theta+1)\frac{(J_{c,p,\lambda}^{m,\delta}f(z))'}{J_{c,p,\lambda}^{m,\delta}L_\theta g(z)}$$

Now dividing both sides by $(p-\gamma)$

$$(d+\frac{\gamma}{(p-\gamma)})(q(p-\eta)+\eta+\theta)+zd'(z)=\frac{z(\theta+1)}{(p-\gamma)}\frac{(J_{c,p,\lambda}^{m,\delta}f(z))'}{J_{c,p,\lambda}^{m,\delta}L_\theta g(z)}$$

Dividing both sides by $(q(p-\eta)+\eta+\theta)$

$$\begin{aligned} d(z)+\frac{zd'(z)}{(q(p-\eta)+\eta+\theta)}&=\frac{1}{(p-\gamma)}\left[\frac{z(\theta+1)(J_{c,p,\lambda}^{m,\delta}f(z))'}{(q(p-\eta)+\eta+\theta)J_{c,p,\lambda}^{m,\delta}L_\theta g(z)}-\gamma\right] \\ d(z)+\frac{zd'(z)}{(q(p-\eta)+\eta+\theta)}&=\frac{1}{(p-\gamma)}\left[\frac{z(J_{c,p,\lambda}^{m,\delta}f(z))'}{J_{c,p,\lambda}^{m,\delta}g(z)}-\gamma\right] \end{aligned}$$

Then from (2.20) and (2.21), we obtain

$$(p-\eta)q+\eta+\theta=\rho e^{\frac{i\pi\phi}{2}}$$

where

$$\left\{\begin{array}{l} \frac{(p-\eta)(1-A)}{(1-B)}+\eta+\theta<\rho<\frac{(p-\eta)(1+A)}{(1+B)}+\eta+\theta \\ -t_2<\phi<t_2 \end{array} \right. \quad \text{for } B \neq -1$$

where t_2 is given by (2.25), and

$$\left\{\begin{array}{l} \frac{(p-\eta)(1-A)}{2}+\eta+\theta<\rho<\infty \\ -1<\phi<1 \end{array} \right. \quad \text{for } B = -1$$

Here, we note that $d(z)$ is analytic in U with $d(0)=1$ in U , by applying the

$$\text{assumption and lemma (2) with } w(z)=\frac{1}{(p-\eta)q+\eta+\theta}$$

Hence $d(z) \neq 0$ in U . If there exist two points $z_1, z_2 \in U$, such that the condition (2.1)

is satisfied then (by lemma 3) , we obtain (2.2) under the restriction(2.3).

At first, for the case $B \neq -1$

$$\arg(d(\mathcal{Z}_1) + \frac{z_1 d'(\mathcal{Z}_1)}{(p-\eta)q(\mathcal{Z}_1)+\eta+\theta}) \leq -\frac{\pi}{2} \alpha_1 - \tan^{-1} \left\{ \frac{(\alpha_1 + \alpha_2)(1 - |\mathcal{C}|^*) \cos \frac{\pi}{2} t_2}{2 \frac{(p-\eta)(1+A)}{(1+B)} + \eta + \theta (1 + |\mathcal{C}|^*) + (\alpha_1 + \alpha_2)(1 - |\mathcal{C}|^*) \sin \frac{\pi}{2} t_2} \right\}$$

$$= -\frac{\pi}{2} \delta_1$$

and

$$\begin{aligned} \arg(d(\mathcal{Z}_2) + \frac{z_2 d'(\mathcal{Z}_2)}{(p-\eta)q(\mathcal{Z}_2)+\eta+\theta}) &\geq \frac{\pi}{2} \alpha_2 + \tan^{-1} \left\{ \frac{(\alpha_1 + \alpha_2)(1 - |\mathcal{C}|^*) \cos \frac{\pi}{2} t_2}{2 \frac{(p-\eta)(1+A)}{(1+B)} + \eta + \theta (1 + |\mathcal{C}|^*) + (\alpha_1 + \alpha_2)(1 - |\mathcal{C}|^*) \sin \frac{\pi}{2} t_2} \right\} \\ &= \frac{\pi}{2} \delta_2 \end{aligned}$$

Where we have used the inequality (2.3), and δ_1, δ_2, t_2 are given by (2.15),(2.16) and (2.25) , respectively.

Similarly , for the case $B = -1$, we obtain

$$\arg(d(\mathcal{Z}_1) + \frac{z_1 d'(\mathcal{Z}_1)}{(p-\eta)q(\mathcal{Z}_1)+\eta+\theta}) \leq -\frac{\pi}{2} \alpha_1$$

and

$$\arg(d(\mathcal{Z}_2) + \frac{z_2 d'(\mathcal{Z}_2)}{(p-\eta)q(\mathcal{Z}_2)+\eta+\theta}) \geq \frac{\pi}{2} \alpha_2$$

These are contradiction to the assumption of theorem (2) .

This completes the proof of theorem (2).

Corollary3. Let $f(z) \in A(p,n)$ and $0 < \delta \leq 1, 0 \leq \gamma < 1$.If

$$\left| \arg \left(\frac{z (J_{c,p,\lambda}^{m,\delta} f(z))'}{J_{c,p,\lambda}^{m,\delta} g(z)} - \gamma \right) \right| < \frac{\pi}{2} \delta$$

For some $g(z) \in S_{c,p,\lambda}^{m,\delta}(\eta, A, B)$

Then

$$\left| \arg \left(\frac{z (J_{c,p,\lambda}^{m,\delta} L_\theta f(z))'}{J_{c,p,\lambda}^{m,\delta} L_\theta g(z)} - \gamma \right) \right| < \frac{\pi}{2} \alpha$$

where $L_\theta(f)$ is defined by (2.8), and $(0 < \alpha \leq 1)$ is the solution of the equation

$$\delta = \begin{cases} \alpha + \frac{2}{\pi} \tan^{-1} \left(\frac{\alpha \cos \frac{\pi}{2} t_2}{\frac{(p-\eta)(1+A)}{1+B} + \eta + \theta + \alpha \sin \frac{\pi}{2} t_2} \right) & \text{for } B \neq -1 \\ \alpha & \text{for } B = -1 \end{cases}$$

where t_2 is given by (2.25)

Proof:

Take $\delta_1 = \delta_2 = \delta$ and $\alpha_1 = \alpha_2 = \alpha$ in theorem(2)

$$\left| \arg \left(\frac{z(J_{c,p,\lambda}^{m,\delta} f'(z))'}{J_{c,p,\lambda}^{m,\delta} g(z)} - \gamma \right) \right| < \frac{\pi}{2} \delta$$

, for some $g(z) \in S_{c,p,\lambda}^{m,\delta}(\eta, A, B)$, then

$$\left| \arg \left(\frac{z(J_{c,p,\lambda}^{m,\delta} L_f'(z))'}{J_{c,p,\lambda}^{m,\delta} L_g(z)} - \gamma \right) \right| < \frac{\pi}{2} \alpha$$

where

$$\delta = \begin{cases} \alpha + \frac{2}{\pi} \tan^{-1} \left(\frac{\alpha \cos \frac{\pi}{2} t_2}{\frac{(p-\eta)(1+A)}{1+B} + \eta + \theta + \alpha \sin \frac{\pi}{2} t_2} \right) & \text{for } B \neq -1 \\ \alpha & \text{for } B = -1 \end{cases}$$

Acknowledgements. The author would like to thank the referee of the paper for his helpful suggestions.

REFERENCES

- [1] F.M, Al-Oboudi, On univalent functions defined by a generalized Salagean operator . Int.J.Math.Math.Sci., **27**(2004),1429-1236.
- [2] S.G.Salagean , Subclasses of univalent functions , Lecture Notes in Math . 1013, Springer-Verlag,Berlin,Heidelberg and New York ,(1983) 362-372.
- [3] M.S.Robertson ,On the theory of univalent functions ,Ann. Math.**37** (1936)374-408.
- [4] P.Enigenberg ,S. S. Miller , P .T . Mocanu and M. O .Reade , On a Briot – Bouquet differential subordination , in : General Inequalities , Vol. 3, Birkhauser , Basel (1983) 339-348.
- [5] M.E . Gordji ,D.Alimohammadi and A.Ebadian,Some inequalities of the generalized Bernardi – Libera-Livingston integral operator on univalent functions .J.Ineq.Pure Appl.Math., **10**(2009), issue 4, article 100.

- [6] S.S.Miller and P.T . Mocanu ,Differential subordinations and univalent functions ,Michigan Math.J.**28**(1981) 157-171.
- [8]M.Nunokawa,S.Owa, H.Saitoh,N .E. Cho and N.Takahashi,Some properties of analytic functions at extremal points for arguments , preprint ,2003.
- [9] Y. Komatu , On analytic prolongation of a family of integral operators . Mathematica (Cluj), **32**(55)(1990) , 141-145 .
- [10] A.Ebadian ,S.Shams, Z.G. Wang and Y . Sun , A class of multivalent analytic functions involving the generalized Jung-Kim-Srivastava operator . Acta Univ. Apulensis , **18**(2009) , 265-277.
- [11] H . Silverman and E .M .Silvia , Subclasses of starlike functions subordinate to convex functions , Canad .J. Math . **37** (1985) 48-61.